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SALARY COMPETITION IN MATCHING MARKETS WITH PRIVATE INFORMATION

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Abstract

We analyze a game in which firms with private information compete for workers by making a single salary offer. Once salaries are chosen, firms make offers to workers, who care only about salary. Firms and workers are matched according to the Gale-Shapley deferred acceptance algorithm that dominates the theory of two-sided matching. For a two-firm, two-worker model, we prove existence of a Bayesian Nash equilibrium in which each firm type chooses a salary according to a continuous distribution with interval support in the salary space. We find a 'separation' of types in equilibrium, in the sense that between two types with a common most preferred worker, one type always makes higher offers than the other type. The type that makes the higher offers depends on the relative marginal values attached to the workers by the different firm types. Moreover, more 'popular' workers attract higher average equilibrium salaries.

We also consider an extension of the model to larger markets by replicating the two-firm, two-worker case, in order to examine the effects of market size on competition and equilibrium salaries. In the limit, there is no aggregate uncertainty about the realization of firm types. We characterize the equilibria for this limit case in which there are a continuum of firms and a continuum of workers divided into two equal-sized worker classes. Finally, we conjecture the existence and convergence of the sequence of equilibria in finite replicated markets to the corresponding continuum equilibrium as the number of replications approaches infinity.

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Salary Competition in Matching Markets with Private Information

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1 Introduction

In competitive labor markets, firms compete with each other along several dimensions. These include salary, employee benefits, bonuses, health insurance coverage, and opportunities for career advancement. Firms often set policies regarding benefits, bonuses, health plans, and vacation time, rather than personalize the terms of employment for each individual worker. It is also common for firms to decide on a salary for a particular position, rather than negotiate a salary with each individual. These terms of employment are often inflexibile, either because they are firm-wide policies, contractual obligations, or because the salary for the position has been widely advertized. There are other settings in which agents on one side of a market make a costly investment in order to compete for the services (or affections) of the agents on the other side of the market. However, in this paper, we will focus on the case of salary competition between firms.

When choosing the terms of employment to offer to workers, each firm considers the preferences (or types) of the other firms. However, in most cases, the firms do not have full information about each other. This paper examines the competitive behavior of firms when they do not know each others' preferences. Although there is a great deal of literature on job matching with salaries when there is complete information, this paper focuses on understanding the effects of private information on salary competition.

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We analyze a game in which firms with private information compete for workers by making a single salary offer. We restrict attention to the case in which each firm wants to hire at most one worker, and each worker can work for only one firm. Firms have preferences (types) over the set of workers, and the option to remain unmatched. We assume that firm types are private information, although there is a common prior distribution over the type space. On the other hand, the workers all rank the firms in order of salary, from highest to lowest.[1] We assume that a firm must commit to a single salary offer, which they are required to pay to the worker with whom they are eventually matched. If they are unmatched, then they do not pay anything. Even though we formulate the problem as one of choosing salaries to offer to the workers, we can also interpret the decisions of the firms as investments in other terms of employment (such as fringe benefits, health plans, or available facilities) that make the firm attractive to potential workers. Under this interpretation, there is less flexibility to personalize the offers made to different workers for the same position.

Once salaries are chosen, the firms and workers are matched in the following manner. Each firm makes an offer to at most one worker. Each worker tentatively accepts at most one offer and rejects all the other offers it receives. Any firm who is rejected then makes the same offer to another worker who has not already rejected the firm. When no new offers are made, all remaining tentative matches are confirmed. This matching process is analogous to the firm-proposing deferred acceptance algorithm introduced by Gale & Shapley (1962) to prove the existence of stable matchings. A nice property of the firm-proposing deferred acceptance algorithm is that it gives firms a dominant strategy to make offers in a straightforward manner. In our setting, it means that firms have a dominant strategy to make offers in order of preference, but only to workers who are acceptable to the firm at its chosen salary. Furthermore, given our assumptions about the preferences of the workers, there is a unique stable matching for any profile of firm preferences and salary offers. As a result, no worker has an incentive to strategically reject an offer. This allows us to focus our attention on analyzing the behavior of the firms when deciding upon their salary offer.

We first consider a two-firm, two-worker model, in which the firms can be one of four types. The first type prefers worker w_1 to worker w_2 whilst the second type only finds worker w_1 to be acceptable. The third type prefers w_2 to w_1 while the fourth type only finds w_2 to be acceptable. Since this is a game of incomplete information, we search for

[1] This is a significant assumption. It eliminates the difficulty of trying to deal with strategic behavior in the offer and matching process, as it induces a unique stable matching.

Bayesian Nash equilibria. We show that there are no pure strategy equilibria. However, we prove the existence of a Bayesian Nash equilibrium in mixed (distributional) strategies which are continuous with interval support.

We find a "separation" of types in equilibrium, in the sense that between two types who have a common most preferred worker, one type always makes higher offers than the other type. For example, in one case, every salary offered by the second type of firm (who only wants worker w_1) is higher than any salary offered by the first type (who also wants w_1 the most, but considers w_2 to be acceptable). That is, all firms of the first type essentially "admit defeat" for the case in which the other firm is of the second type. Instead, firms of the first type concentrate on competing against the realization of another firm of the first type. We show that the relative marginal value attached to the workers determines which type makes the higher offers in equilibrium. When the realized types do not share a common most preferred type, there is no competitive pressure on the salary offers. Therefore, the relative probabilities of being one of the first two types, compared with the third or fourth types, also influences the equilibrium salaries. We show that more popular workers (as determined by the distribution over firm types), attract higher average salaries. Likewise, as one might expect, we show that the higher the probability of facing a given firm type, the higher the average salary offered by that type of firm in equilibrium.

We then extend the analysis to larger markets by replicating the two-firm, two-worker market. We prove the existence of a Bayesian Nash equilibrium in continuous distributional strategies with interval support for each finite replicated market. The proof, which is by construction, also establishes the 'separation' result for types with a common most preferred worker class. We also characterize the equilibrium strategies for the limit case in which there are a continuum of firms and a continuum of workers. Finally, we show numerically that the finite market equilibrium strategies converge to the corresponding continuum equilibrium strategies as the number of replications approaches infinity.

Section 2 discusses the existing literature on job matching with salaries. The key observation is that all of this literature assumes that firms have complete information. In Section 3, we describe the model and the matching process, then introduce the general two-firm, two-worker game with private information. We construct a simple example to demonstrate the main features of the equilibrium and identify some of the intuition behind the behavior of the firms. We then characterize the symmetric Bayesian Nash equilibria for which strategies are continuous distributions with interval support. In particular, we prove the 'separation' result described above for a general two-firm, two-worker model with private information.

In Section 4, we introduce the extension to larger markets. We build larger markets while maintaining a manageable type space by replicating the two-firm, two-worker baseline model. Before discussing the existence and properties of the equilibria for finitely replicated markets, we characterize the equilibria of the limit case in which there are a continuum of firms and a continuum of workers. The equilibria for the continuum case can be derived more easily than for finite markets because there is no aggregate uncertainty when there are infinitely many firms and workers. Nevertheless, we prove existence of equilibria for finite replicated markets and then show numerically, that the equilibrium strategies for the different firm types converge to the corresponding continuum equilibrium as the number of replications approaches infinity.

2 Related Literature

Our work is closely related to a number of other papers that deal with matching firms to workers by incorporating salary offers. Perhaps the earliest treatment of matching with salaries is the work by Shapley & Shubik (1972), who modify the work of Gale & Shapley (1962) to incorporate a transferrable utility good in which salaries can be paid. They used a linear programming duality approach to prove that there exists a core allocation, or a one-to-one matching along with a salary schedule, in which no firm and no worker can negotiate a salary at which they would prefer each other over their current partners at their current salaries. This work was further developed by Crawford & Knoer (1981), and subsequently, by Kelso & Crawford (1982), who devised a salary adjustment process which converges to a core allocation, as described above. According to this process, firms propose to their favorite worker, based on a given salary schedule. Workers reject all but their favorite offer, and the rejected firms then update the salary schedule, by increasing the salary to be offered to workers who have rejected them, and make new offers. The process ends when there are no rejections issued in some step. More recently, Hatfield & Milgrom (2005) have developed a model of matching with contracts that incorporates the Kelso-Crawford model and ascending package auction models. They show that if the preferences of the firms satisfy a *substitutes* condition and a *law of aggregate demand* condition, then truthful reporting is a dominant strategy for workers in a worker-proposing matching mechanism.

Bulow & Levin (2006), in response to an antitrust case brought against the National Resident Matching Program (NRMP), consider the effect of a centralized matching mechanism on salary levels. Their paper argues that the NRMP algorithm compresses and depresses the salaries of the workers, relative to the competitive equilibrium salaries. Their paper, along with the debate about the effects of the algorithm, has attracted further work on the issue. Niederle (2007) studies a special feature of the NRMP algorithm that allows firms to make "ordered contracts" which support an equilibrium in which salaries are competitive. Furthermore, Kojima (2007) shows that if the Bulow & Levin model is extended to allow for firms with more than one available position, then the centralized matching algorithm may actually increase salaries relative to competitive equilibrium. The glaring absence from the existing literature on matching with salaries is a model in which the firms may have preferences that are private information. All of the models that have been developed to explain the matching process when firms can make offers with salaries attached, assume that the preferences are complete information. In this paper, we provide the first step towards a general model of matching with salaries under private information.

Our model is most similar to the Bulow & Levin setup. However, our models differ in two main ways. Perhaps most importantly, we allow the firms to have different primitive preferences over the workers, whereas Bulow & Levin assume that all firms rank workers in the same way, according to their publicly known productivity. By allowing for firms to have different preferences over the workers, we are able to relax the assumption of complete information. Instead, we suppose that each firm's type is private information, drawn independently from a commonly known distribution over the type space. This is where our model departs from the existing literature on matching with salaries.

3 A Two-Firm, Two-Worker Model

Suppose there are two firms $f_1, f_2 \in F$ and two workers $w_1, w_2 \in W$. Each firm has strict preferences over the set $\{w_1, w_2, \emptyset\}$, where $\emptyset$ represents being unmatched. We ignore any preference ranking in which remaining single is the most preferred option. Thus, there are four possible preference rankings for each firm.

$$P_a : \; w_1 \, w_2 \, \emptyset \qquad P_b : \; w_1 \, \emptyset \, w_2$$
$$P_c : \; w_2 \, w_1 \, \emptyset \qquad P_d : \; w_2 \, \emptyset \, w_1.$$

We assume that each preference ranking is represented by a pair of values, one for each worker, while the value of remaining unmatched is 0. This assumption is somewhat restrictive, since it means that two firms with same preference ranking also have the same values for the workers. In Section 5, I discuss ways to relax this assumption about the type space.

We refer to a firm with preferences P_k as a firm of type k. Then the set of firm types is described as $\mathcal{P}^f = \{a, b, c, d\}$ where, for example, $a = (a_1, a_2)$ and a_j is the value of worker j to type a for each $j = 1, 2$. In order to represent the preference rankings, we have the following restrictions on the values of the different types.

$$\begin{aligned} &a_1 > a_2 > 0 \\ &b_1 > 0 > b_2 \\ &c_2 > c_1 > 0 \\ &d_2 > 0 > d_1 \quad . \end{aligned}$$

Definition 1. *A worker w is* **acceptable** *to firm f if f prefers w to remaining unmatched.*

We modify this standard notion of an acceptable worker to account for the preferences of the firms at a given salary level.

Definition 2. *Given any salary, x_f, chosen by firm f, a worker w is* **salary-acceptable** *to firm f if f's value for worker w is greater than x_f.*

The values corresponding to each type are common knowledge, however, each firm knows only its own type. The types are drawn independently according to the common prior distribution π over $\mathcal{P}^f = \{a, b, c, d\}$. Given the two disjoint sets of agents, we define a matching as follows.

Definition 3. *A matching is a function $\mu : F \cup W \to F \cup W \cup \emptyset$ such that*

(1) $\mu(f) \in W \cup \emptyset$ for all $f \in F$,

(2) $\mu(w) \in F \cup \emptyset$ for all $w \in W$, and

(3) $\mu(\mu(i)) = i$ for all $i \in F \cup W$ with $\mu(i) \neq \emptyset$.

We let $\mathcal{M}$ denote the set of all matchings.

For any firm f with type $k = (k_1, k_2)$, the utility derived from a matching $\mu \in \mathcal{M}$ is given by

$$u_k^f(\mu) \quad = \quad \begin{cases} k_1 & \text{if } \mu(f) = w_1 \\ k_2 & \text{if } \mu(f) = w_2 \\ 0 & \text{if } \mu(f) = \emptyset \end{cases} \quad .$$

Before the matching is determined, the firms each choose a salary. Then the following steps determine the matching outcome.

Step 1: Each firm makes an offer to (at most) one worker;

Step 2: Each worker tentatively accepts at most one offer, and rejects all others;

$\vdots$

Step k**:** Any firm whose most recent offer was rejected may make the same salary offer to a worker who has not already rejected them;

Step $k+1$**:** Each worker tentatively accepts at most one offer out of the one (if any) it tentatively holds, and the new offers received at Step k, and rejects all others.

The procedure terminates when no new offers are made, and then all tentative matches are confirmed.

In principle, both the firms and workers could adopt a large number of different strategies, some of which may be incredibly complex. Fortunately, we do not need to consider every strategy, as the following two remarks make clear.

Remark 1. *For any set of chosen salaries, each firm has a dominant strategy to make offers in order of preference to salary-acceptable workers only.*

Once firms have chosen salaries, the matching procedure described above is equivalent to the Gale-Shapley Deferred Acceptance (DA) algorithm for a very particular matching market. The relevant matching market is the one in which the firms' preferences are their original preferences, restricted to their (respective) sets of salary-acceptable workers, and workers' preferences are given by ranking the firms according to salary, from highest to lowest. It follows from Theorem 5 in Roth (1982), that firms have a dominant strategy to make offers in order of preferences, but restricted to salary-acceptable workers.

Remark 2. *For any profile of firm preferences and any set of chosen salaries, each worker has a dominant strategy to reject all but the highest salary offered to them.*

Recall that the firm-proposing DA mechanism is stable. That is, for any profile of reported preferences, it produces a matching that is stable with respect to the reported preferences. Since all the workers have the same preferences, there is a unique stable matching for each realization of firm preferences and set of chosen salaries. By Theorem 4.16 in Roth & Sotomayor (2006), every set of worker strategies that form a Nash equilibrium with the truthful strategies of the firms produces a matching that is stable with respect to the true preferences. Then since each induced market has a unique stable matching, and the matching mechanism is stable, there is no other Nash equilibrium strategy that dominates truth-telling by the workers, for any realization of firm types.

Since there are no incentives for strategic sequencing of offers by the firms, or strategic rejection by the workers, we focus our attention on the behavior of the firms when they decide upon a salary. We can describe the outcomes from the matching process outlined above by a direct revelation outcome function g. Let $g : \mathcal{P} \times \mathbb{R}_+^2 \to \mathcal{M}$ be an outcome function that maps the preferences (types) of the two firms and the salaries chosen by the firms into the set of matchings. As we noted above, the firms have a dominant strategy to announce their true preferences over salary-acceptable workers and the workers simply reject all but the highest offer made to them.

3.1 Pure Strategy Equilibria

Consider the game $\Gamma = \left(F, W, \mathcal{P}, \mathbb{R}_+, \pi, \mathbf{g}, \{u_k^f\}_{f,k}\right)$. We have the set of firms F, workers W, the firm type space $\mathcal{P}$, the space of possible salaries $\mathbb{R}_+$, and the type distribution π. The outcome function $\mathbf{g}$ represents the matching process described above, and $\{u_k^f\}_{f,k}$ are the utility functions for each firm and each firm type over the set of matchings.

A pure strategy for a firm f is a function $\mathbf{s}_f : \mathcal{P}^f \to \mathbb{R}_+$ which selects a salary for each possible firm type. Given a strategy $\mathbf{s}_{-f}$ for the other firm, firm f's expected payoff from announcing a salary x_f when its type is k is given by

$$\mathbb{E}U_k^f\left(x_f, \mathbf{s}_{-f},\right) = \sum_{p \in \mathcal{P}^{-f}} \pi(p) \cdot u_k^f \left[g\left(k, p, x_f, s_{-f}(p)\right)\right].$$

We argue that there is no pure strategy Bayesian Nash equilibrium to this game.

Consider any arbitrary pair of strategies $(\mathbf{s}_1, \mathbf{s}_2)$ and suppose firm 1's type is a. Notice that, if firm 2's type is either of c or d, then regardless of $s_1(a)$, firm 1 is matched with worker w_1. However, if firm 2's type is a or b, then the outcome depends on the salaries announced by the firms.

If firm 2 is playing $\mathbf{s}_2$, then firm 1's best response is to announce $s_1(a) = max\{s_2(a), s_2(b)\} + \varepsilon$ as long as $s_1(a) \leq a_1 - a_2$. If $max\{s_2(a), s_2(b)\} \geq a_1 - a_2$, then firm 1's best response is to announce $s_1(a) = 0$. However, given the choice of firm 1, $s_1(a) = max\{s_2(a), s_2(b)\} + \varepsilon$, firm 2's best response, if it is type a, is to offer $s_2(a) = s_1(a) + \varepsilon$, up to $s_2(a) \leq a_1 - a_2$. The same type of 'incremental' best responses exist for type b firms, and by symmetry, also for types c and d.

The problem with pure strategies is that firms who have a common most preferred worker will continue to outbid each other until the marginal benefit of 'winning' the worker is equal to the marginal benefit of not winning. However, once that point is reached, the best response is to announce a salary of 0, and begin the upbidding process all over again. Instead we look for equilibria in mixed strategies. Given the symmetric nature of the game, we search for a symmetric equilibrium.

3.2 Mixed (Distributional) Strategy Equilibria

Formally, a mixed strategy for firm i is a function $\sigma_i : \mathcal{P} \to \Delta(\mathbb{R}_+)$ which announces, for each preference type, a distribution over salaries in $\mathbb{R}_+$. We will refer to the symmetric equilibrium (σ^*, σ^*) by the equilibrium strategy $\sigma^* = (G^*_a, G^*_b, G^*_c, G^*_d)$ where G^*_k is the cumulative distribution announced by the firms when their type is k. We assume that strategies are continuous distributions with interval support.[2] We establish the existence of a symmetric Bayesian Nash equilibrium in distributional strategies that are continuous with interval support.

Before we prove any results, we provide a simple example for the two-firm, two-worker model.

[2]There may be other types of symmetric equilibria, with non-interval support, or discontinuous strategies. In addition, there may be asymmetric equilibria.

Example 1. *Suppose $a = (2, 1)$, $b = (2, -1)$, $c = (1, 2)$, and $d = (-1, 2)$, while $\pi(a) = \frac{1}{2}$, $\pi(b) = \frac{1}{8}$, $\pi(c) = \frac{1}{4}$, and $\pi(d) = \frac{1}{8}$. Notice that the marginal benefit to getting worker w_1 is higher for type b than type a, and the marginal benefit to getting worker w_2 is higher for type d than type c. Given these parameters, we conjecture that type b firms will make higher offers than type a firms, and type d firms will make higher offers than type c firms. Furthermore, given the distribution of types π, worker w_1 is in a sense more popular than w_2. As such, we might expect to see higher salaries on average being offered to w_1.*

We find an equilibrium described as follows:

$$\begin{aligned}
G_a^*(x) &= 2x \quad \textit{on the support } \left[0, \frac{1}{2}\right] \\
G_b^*(x) &= \frac{7x - 3.5}{2 - x} \quad \textit{on the support } \left[\frac{1}{2}, \frac{11}{16}\right] \\
G_c^*(x) &= 4x \quad \textit{on the support } \left[0, \frac{1}{4}\right] \\
G_d^*(x) &= \frac{7x - 1.75}{2 - x} \quad \textit{on the support } \left[\frac{1}{4}, \frac{15}{32}\right].
\end{aligned}$$

There are several interesting features exhibited by this equilibrium. One observation is that there is no overlap between the equilibrium supports of types with a common most preferred worker. Since the marginal value of getting worker w_1 is less for type a than for type b, firms of type b **always** announce higher salaries than firms of type a. In other words, firms of type a are resigned to getting their second favorite worker (w_2) when the other firm is type b.

Instead, a type a firm focuses just on competing against another type a firm. In contrast, a type b firm offers enough to ensure that it outbids any type a firm, then competes against the chance that the other firm is also type b. This type of 'separation' result between types a and b is also exhibited by types c and d, and we show below that it is a characteristic of any equilibrium in continuous distributional strategies with interval support.

Another feature of the equilibrium for this example is that salaries are higher on average for firms of type a than type c and for type b than type d, even though they have comparable values for their respective preferences. This reflects the relative 'popularity' of worker w_1 relative to worker w_2. This notion of popularity is manifested in the differences in the probabilities of facing another firm with the same most preferred worker. For types

a and b, the probability of facing another type a or b is $\frac{5}{8}$, while for types c and d, the probability of facing another type c or d is only $\frac{3}{8}$. As a result, the average salaries offered in equilibrium are higher for type a than type c, and higher for type b than type d. Below, we show that this feature is a general result that applies to all equilibria of the game.

Consider again the general model. Note that the strategies of types that share a common most preferred worker affect each other. On the other hand, salaries do not affect the matching output when the realized types do not have a common most preferred worker. Thus, we can consider pairs of types in isolation from one another. Without loss of generality, we consider types a and b. The following two lemmas allow us to characterize the supports for the equilibrium strategies.

Lemma 1. *Between types with a common most preferred worker, the lowest salary offered in equilibrium must be 0.*

Proof

Let $[\underline{x}_a, \overline{x}_a]$ and $[\underline{x}_b, \overline{x}_b]$ be the equilibrium supports for types a and b respectively. Suppose by means of contradiction that neither $\underline{x}_a$ nor $\underline{x}_b$ is equal to 0. Consider $0 < \underline{x}_a \leq \underline{x}_b$. Type a's expected payoff from $x = \underline{x}_a$ is

$$\mathbb{E}U_a(\underline{x}_a) = [\pi(a) + \pi(b)]a_2 + [\pi(c) + \pi(d)]a_1 - \underline{x}_a$$

and for any $x \in [0, \underline{x}_a)$, type a's expected payoff is

$$\begin{aligned} \mathbb{E}U_a(x) &= [\pi(a) + \pi(b)]a_2 + [\pi(c) + \pi(d)]a_1 - x \\ &< \mathbb{E}U_a(\underline{x}_a). \end{aligned}$$

This means that $[\underline{x}_a, \overline{x}_a]$ cannot be an equilibrium support unless $\underline{x}_a = 0$ or $0 \leq \underline{x}_b < \underline{x}_a$.

If $0 < \underline{x}_b \leq \underline{x}_a$, type b's expected payoff from $x = \underline{x}_b$ is

$$\mathbb{E}U_b(\underline{x}_b) = (b_1 - \underline{x}_b)[\pi(c) + \pi(d)].$$

That is, at the lower bound of type b's equilibrium support, a firm of type b does not get matched to a worker unless the other firm is type c or type d. But in those cases, the salary does not affect the outcome, so that choosing a salary of $\underline{x}_b > 0$ is strictly dominated by $x = 0$. Thus, $[\underline{x}_b, \overline{x}_b]$ cannot be an equilibrium support unless $\underline{x}_b = 0$ or $0 \leq \underline{x}_a < \underline{x}_b$. Therefore, in equilibrium, we must have either $\underline{x}_a = 0$ or $\underline{x}_b = 0$. ■

A similar proof technique helps to prove the next lemma, which further restricts the type of supports that we can expect in equilibrium.

Lemma 2. *In equilibrium, there are no gaps between the equilibrium supports for types with a common most preferred worker.*

Proof

Suppose $\overline{x}_a < \underline{x}_b$. Then $\forall x \in (\overline{x}_a, \underline{x}_b)$, type b's expected payoff is

$$\begin{aligned} \mathbb{E}U_b(x) &= (b_1 - x)(1 - \pi(b)) \\ &> (b_1 - \underline{x}_b)(1 - \pi(b)) = \mathbb{E}U_b(\underline{x}_b), \end{aligned}$$

contradicting the inclusion of $\underline{x}_b$ in the equilibrium support for type b. The proof is similar for the case when $\overline{x}_b < \underline{x}_a$. Since the supports are intervals by assumption, there are no other cases to be considered. ■

These two lemmas imply that equilibria must be consistent with one of four cases. In each case, type a mixes over $[\underline{x}_a, \overline{x}_a]$, and type b mixes over $[\underline{x}_b, \overline{x}_b]$, where

Case 1: $0 = \underline{x}_a < \underline{x}_b \leq \overline{x}_a < \overline{x}_b$

Case 2: $0 = \underline{x}_b < \underline{x}_a \leq \overline{x}_b < \overline{x}_a$

Case 3: $[\underline{x}_a, \overline{x}_a] \subset [\underline{x}_b, \overline{x}_b]$, and $\underline{x}_b = 0$

Case 4: $[\underline{x}_b, \overline{x}_b] \subset [\underline{x}_a, \overline{x}_a]$, and $\underline{x}_a = 0$.

The proposition below generalizes and formalizes the 'separation' result we found for the equilibrium in the example, and shows that there are no equilibria of the form described by **Case 3** or **Case 4**.

Proposition 1. *Equilibrium supports do not overlap for types with a common most preferred worker. In particular then, all equilibria must be of the form in* **Case 1** *with* $\underline{x}_b = \overline{x}_a$ *or* **Case 2** *with* $\underline{x}_a = \overline{x}_b$.

Proof

See Appendix. ■

The proof for Proposition 1 is based on demonstrating that we cannot simultaneously satisfy indifference for both types on an interval with non-empty interior. As a result, the equilibrium supports in **Case 1** and **Case 2** must meet at their boundaries. For **Case 3** and **Case 4**, the same argument implies that the support which is a subset of the other must be a single point. However, since we have already ruled out best responses in pure strategies, there cannot exist an equilibrium in either of these cases.

The next proposition characterizes all of the symmetric Bayesian Nash equilibria in which strategies are continuous with interval support. Moreover, it provides the set of conditions that determine, for each pair of types with a common most preferred worker, whether their equilibrium supports are consistent with **Case 1** or **Case 2**. The condition depends on the relative marginal benefits of getting the types' common most preferred worker, and on the probability that a firm is the type that also finds the other worker acceptable.

Proposition 2. *If $b_1 > \pi(a)(a_1 - a_2)$, then in all equilibria,*

$$\begin{aligned} G_a^*(x) &= \frac{x}{\pi(a)(a_1 - a_2)} \\ &\quad \textit{on the support } \Big[0,\, \pi(a)(a_1 - a_2)\Big] \\ G_b^*(x) &= \frac{1 - \pi(b)}{\pi(b)(b_1 - x)}\left[x - \pi(a)(a_1 - a_2)\right] \\ &\quad \textit{on the support } \Big[\pi(a)(a_1 - a_2),\, \pi(b)b_1 + (1 - \pi(b))\pi(a)(a_1 - a_2)\Big], \end{aligned}$$

regardless of G_c^*, G_d^*. The analogous result holds for types c and d if $d_2 > \pi(c)(c_2 - c_1)$.

If $b_1 < \pi(a)(a_1 - a_2)$, *then in all equilibria,*

$$\begin{aligned} G_b^*(x) &= \frac{x(\pi(c)+\pi(d))}{\pi(b)(b_1 - x)} \\ &\quad \textit{on the support } \left[0, \frac{\pi(b)b_1}{1-\pi(a)}\right] \\ G_a^*(x) &= \frac{(1-\pi(a))x - \pi(b)b_1}{\pi(a)(1-\pi(a))(a_1 - a_2)} \\ &\quad \textit{on the support } \left[\frac{\pi(b)b_1}{1-\pi(a)}, \frac{\pi(b)b_1}{1-\pi(a)} + \pi(a)(a_1 - a_2)\right]. \end{aligned}$$

The analogous result holds for types c *and* d *if* $d_2 < \pi(c)(c_2 - c_1)$.

Proof

See Appendix. ■

Part of the condition in Proposition 2 has a simple intuition. If type b gets a higher value from worker w_1 than the marginal value for type a from getting w_1 instead of w_2, then type b will be willing to pay more than type a for w_1. The role of $\pi(a)$ in the condition is less obvious. Keeping the values fixed, if $\pi(a)$ is relatively low, a type a firm does not need to mix over a large interval to compete against its own type. As a result, if type a firms offer salaries above those offered by type b, there may be an incentive for type b firms to offer salaries higher than the type a firms in order to 'steal' worker w_1 in the event that the other firm is type a. Any such deviation by type b firms would give type a firms an incentive to lower the support of their distributional strategies to a lower bound of 0.

The following two corollaries of Proposition 2 confirm two expected features of the equilibrium. All things being equal, the more likely a firm is to face another firm of the same type, the stronger the competitive pressure and the higher the average equilibrium salary for their firm type. Similarly, the more likely a firm is to face another firm with the same most preferred worker, the stronger the competition and the higher the average equilibrium salary offered by the two relevant firm types.

Corollary 1. *The higher the probability a firm type has to compete against its own type, the higher (on average) the equilibrium salary offered by that firm type.*

Proof

In **Case 1**, the expected salary offer of firm type a is just the expected value of a uniform random variable on $\Big[0, \pi(a)(a_1 - a_2)\Big]$ - that is,

$$\mathbb{E}(x_a) = \frac{\pi(a)(a_1 - a_2)}{2},$$

which is strictly increasing in $\pi(a)$. For type b, the expected salary is

$$\mathbb{E}(x_b) = \int\limits_{\pi(a)(a_1-a_2)}^{\pi(b)b_1+(1-\pi(b))\pi(a)(a_1-a_2)} x \cdot g_b^*(x)dx. \tag{1}$$

Recall that the distribution

$$G_b^*(x) \quad = \quad \frac{1-\pi(b)}{\pi(b)(b_1 - x)} \left[x - \pi(a)(a_1 - a_2)\right],$$

which gives

$$g_b^*(x) = \frac{1-\pi(b)}{\pi(b)} \left(\frac{b_1 - \pi(a)(a_1 - a_2)}{(b_1 - x)^2} \right). \tag{2}$$

Since $b_1 > \pi(a)(a_1 - a_2)$ for this case, we have $b_1 > x$, and $g_b^*(x)$ is increasing in x.

Furthermore,

$$\frac{\partial g_b^*(x)}{\partial \pi(b)} = -\frac{b_1 - \pi(a)(a_1 - a_2)}{[\pi(b)(b_1 - x)]^2} < 0 \tag{3}$$

implies that $g_b^*(x)$ decreases as $\pi(b)$ increases. However, $\frac{\partial^2 g_b^*(x)}{\partial \pi(b)^2}$ is also negative, which means that the decrease in $g_b^*(x)$ from an increase in $\pi(b)$ is more severe for lower values of x.

Since the upper bound of the integration is increasing in $\pi(b)$, the expected value of the salary offered by type b must be increasing with $\pi(b)$, since we assign positive weight to higher salaries not previously included, and the weight attached to those salaries that were previously included falls more for lower salaries than higher salaries.

The proof for Case 2 is a similar series of calculations to verify that the expected salary is increasing in the probability of the firm type. We should also mention that, for some values of b_1 and $a_1 - a_2$, as $\pi(a)$ is increasing, it may cause the equilibrium to switch from **Case 1** to **Case 2**. In this case, there is some chance that the average equilibrium salary will jump down (or up), however, for any further increases in $\pi(a)$, the result will continue to hold. ■

Corollary 2. *For any firm, the higher the probability that the other firm has the same most preferred worker, the higher the equilibrium salary (on average) offered by the firm.*

Proof

We show that the average equilibrium salary for a given firm type is non-decreasing in the probability of the other firm type with the same most preferred worker. Together with Corollary 1, this implies the result. For **Case 1**, type a's expected salary does not depend on $\pi(b)$. On the other hand, for type b, both the lower and upper bounds of the support increase with $\pi(a)$. Futhermore, $g_b^*(x)$ is decreasing in $\pi(a)$, but does so more severely for lower salaries. Therefore, the expected salary for type b increases with $\pi(a)$.

Again, we follow the same steps for proving the result in **Case 2**, and show that in that case, the expected salary for type a is actually increasing in $\pi(b)$. We also reiterate the caveat that for some values of b_1 and $a_1 - a_2$, an increase in $\pi(a)$ may cause the equilibrium to switch from **Case 1** to **Case 2**. For type b, this means that the expected salary ought to jump down discretely, lowering the expected salaries of type b for high enough values of $\pi(a)$. Nevertheless, within a particular case, the expected salary for type b is increasing in $\pi(a)$. ■

4 Competition in Replicated Markets

In this section, we examine equilibrium behavior in large markets. For tractability, we replicate the two-firm, two-worker market to obtain a market with $2n$ firms and $2n$ workers, consisting of n identical "class w_1" workers and n identical "class w_2" workers. First, we characterize the equilibria for the case in which there are a continuum of firms and a continuum of workers. Then we prove existence, for finite replicated markets, of a Bayesian Nash equilibrium in continuous distributional strategies with interval supports. We show that both Proposition 1 and Proposition 2 generalize to replicated markets.

Finally, we show numerically that the limit of the sequence of replicated market equilibria as the number of replications approaches infinity is the corresponding equilibrium for the market with a continuum of agents. This suggests that replication reduces aggregate uncertainty about the realization of types among the firms. As a result, salary competition increases with replication for the more popular worker class, while it eventually disappears for the less popular class.

4.1 Market Replication

In order to avoid the complexity of dealing with an exponentially growing number of firm types, we analyze a baseline model with two distinct workers who are replicated to form larger markets with many firms. This provides a convenient way to conduct a tractable analysis of competitive behavior in large markets. Furthermore, it removes the chance of realizing an uninteresting market with sparse competition, in which every firm wants a different type of worker.

To be more specific, the baseline market is one with two firms, $F^1 = \{f_1, f_2\}$ and two distinct workers $W = \{w_1, w_2\}$. In an n-replicated market, there are $2n$ firms, $F^n = \{f_1, ..., f_{2n}\}$, along with n copies of w_1, $W_1 = \{w_1^1, w_1^2, ..., w_1^n\}$, and n copies of w_2, $W_2 = \{w_2^1, ..., w_2^n\}$. Since w_1^j and w_1^k are identical copies of one another, we assume that all firms are indifferent between any two workers in W_1. Likewise, all firms are indifferent between any two workers in W_2. As a result, we can define firms' preferences (and from these, their types) as strict orderings over the set $\{W_1, W_2, \emptyset\}$.

As in section 3, we rule out the possibility of firm types that prefer being unmatched over every worker.[3] So we are left with four possible firm types that are essentially the same as the types in the two-firm, two-worker case, except that the preferences are over classes of workers W_1 and W_2.

$$\begin{array}{ll} P_a : W_1\, W_2\, \emptyset & \qquad P_b : W_1\, \emptyset\, W_2 \\ P_c : W_2\, W_1\, \emptyset & \qquad P_d : W_2\, \emptyset\, W_1. \end{array}$$

We again assume that each preference ranking is represented by a pair of values - one for each worker class, W_1 and W_2 - while the value of remaining unmatched is normalized to 0. So, for each type $k \in \{a, b, c, d\}$, we have $k = (k_1, k_2)$, where k_i is the value of each worker w in the class W_i. For the values to represent the corresponding preference

[3] We may just as well assume that they don't enter the market in the first place.

rankings, they must satisfy

$$
\begin{aligned}
a_1 &> a_2 > 0 \\
b_1 &> 0 > b_2 \\
c_2 &> c_1 > 0 \\
d_2 &> 0 > d_1 \quad .
\end{aligned}
$$

Each firm knows only its own type, and the types are drawn independently according to the common prior distribution π over $\{a, b, c, d\}$. That is, $\pi(k)$ is the probability that a given firm is a type k firm, or equivalently, has preferences P_k.

4.2 Equilibrium in the Continuum Case

Before analyzing the equilibria for a finite replicated market, we examine the equilibrium behavior in the limit, when there is a continuum of firms, and continuum of workers, such that the measure of workers in each class W_1 and W_2 is half the total measure of W. In this environment, since there are infinitely many firms, the aggregate uncertainty about the realized firm types disappears from the market. That is, $\pi(k)$ is the actual proportion, or the measure of type k firms in the market. This is a convenient feature because it makes the equilibrium strategies relatively straightforward functions of the distribution π.

As for the two-firm, two-worker case, we can establish that the equilibrium strategy for a given type k does not depend on the strategies of the two types k', k'' that have a different most preferred worker class than type k. Thus, as in section 3, when deriving equilibrium strategies, we can deal with types a and b independently from types c and d.

We consider only types a and b, since the analysis will be symmetric for types c and d. The following proposition provides a characterization of the equilibria. The proposition is broken into two cases based on the relative marginal values for types a and b of worker class W_1 over W_2.

Proposition 3. *The following two cases characterize the equilibria when there are a continuum of firms and a continuum of workers, with two equally large worker classes.*

Case 1: $b_1 \geq a_1 - a_2$

- *If* $\pi(a)+\pi(b) \leq \frac{1}{2}$, *then* $x_a^* = 0$ *and* $x_b^* = 0$.
- *If* $\pi(a) > \frac{1}{2}$, *then* $x_b^* = a_1 - a_2$ *and*

$$x_a^* = \begin{cases} 0 & \text{with probability } p_a(0) = \frac{2(\pi(a)+\pi(b))-1}{2\pi(a)} \\ a_1 - a_2 & \text{with probability } 1 - p_a(0) \end{cases}.$$

- *If* $\pi(b) > \frac{1}{2}$, *then* $x_a^* = 0$ *and* $x_b^* = b_1$.
- *If* $\pi(a) \leq \frac{1}{2}$, $\pi(b) \leq \frac{1}{2}$, *but* $\pi(a)+\pi(b) > \frac{1}{2}$, *then* $x_b^* = a_1 - a_2$ *and*

$$x_a^* = \begin{cases} 0 & \text{with probability } p_a(0) = \frac{2(\pi(a)+\pi(b))-1}{2\pi(a)} \\ a_1 - a_2 & \text{with probability } 1 - p_a(0) \end{cases}.$$

Case 2: $b_1 < a_1 - a_2$

- *If* $\pi(a)+\pi(b) \leq \frac{1}{2}$, *then* $x_a^* = 0$ *and* $x_b^* = 0$.
- *If* $\pi(a) > \frac{1}{2}$, *then* $x_b^* \in [0, b_1]$ *and*

$$x_a^* = \begin{cases} 0 & \text{with probability } q_a(0) = \frac{2\pi(a)-1}{2\pi(a)} \\ a_1 - a_2 & \text{with probability } 1 - q_a(0) \end{cases}.$$

- *If* $\pi(b) > \frac{1}{2}$, *then* $x_a^* = b_1$ *and* $x_b^* = b_1$.
- *If* $\pi(a) \leq \frac{1}{2}$, $\pi(b) \leq \frac{1}{2}$, *but* $\pi(a)+\pi(b) > \frac{1}{2}$, *then* $x_a^* = b_1$ *and* $x_b^* = b_1$.

Figures 1 and 2 provide graphical illustrations of the two cases in Proposition 3. Each figure plots $\pi(a)$ against $\pi(b)$ and divides the space of probability pairs $(\pi(b), \pi(a))$ into segments for each subcase of the equilibrium characterization. In both Figure 1 and Figure 2, the bottom left triangle corresponds to the case in which there is an excess supply of class W_1 workers, and therefore no competition between types a and b. Thus, $x_a^* = x_b^* = 0$ for both cases when $\pi(a)+\pi(b) \leq \frac{1}{2}$.

In Figure 1, we can merge the subcase in which $\pi(a) > \frac{1}{2}$ with the subcase in which $\pi(a) \leq \frac{1}{2}$ and $\pi(b) \leq \frac{1}{2}$, but $\pi(a)+\pi(b) > \frac{1}{2}$, since in each subcase, type a firms mix between 0 and $a_1 - a_2$ with probability $p_a(0) = \frac{2[\pi(a)+\pi(b)]-1}{2\pi(a)}$, while type b firms choose $a_1 - a_2$. Finally, in the case when $\pi(b) > \frac{1}{2}$, type b firms compete with each other and push the salary up to their marginal value from a class W_1 worker, while type a firms know that they will not be matched with a class W_1 worker and so choose a salary of 0.

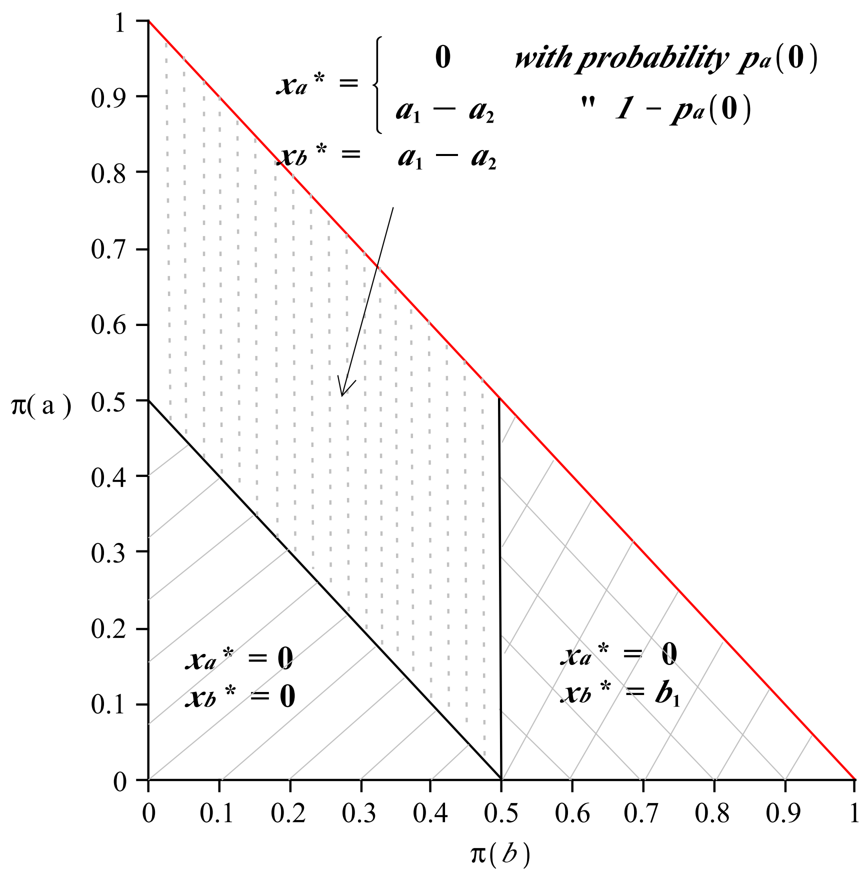

Figure 1: Continuum Equilibria for $b_1 \geq a_1 - a_2$

In Figure 2, we can likewise merge the subcase in which $\pi(b) > \frac{1}{2}$ with the subcase in which $\pi(a) \leq \frac{1}{2}$ and $\pi(b) \leq \frac{1}{2}$, but $\pi(a) + \pi(b) > \frac{1}{2}$, since in each subcase, both type a firms and type b firms choose a salary of b_1. When $\pi(a) > \frac{1}{2}$, type a firms mix between 0 and $a_1 - a_2$ with probability $q_a(0) = \frac{2\pi(a)-1}{2\pi(a)}$, while type b firms choose a salary in the interval $[0, b_1]$. This is because type a firms drive the salary for a class W_1 worker up to $a_1 - a_2 > b_1$, so that type b firms are never matched with anyone. Since some of the type a firms will miss out on a class W_1 worker, they mix between the salary $a_1 - a_2$ and 0.

From Proposition 3, given any set of values for the types and a type distribution π, we can determine the continuum equilibrium. This provides us with a benchmark for the limit case. In the next subsection, we consider the conjecture that the sequence of equilibria in n-replicated markets converge to the relevant continuum equilibrium as n approaches infinity. Since the result is not yet proved, we just mention the main elements

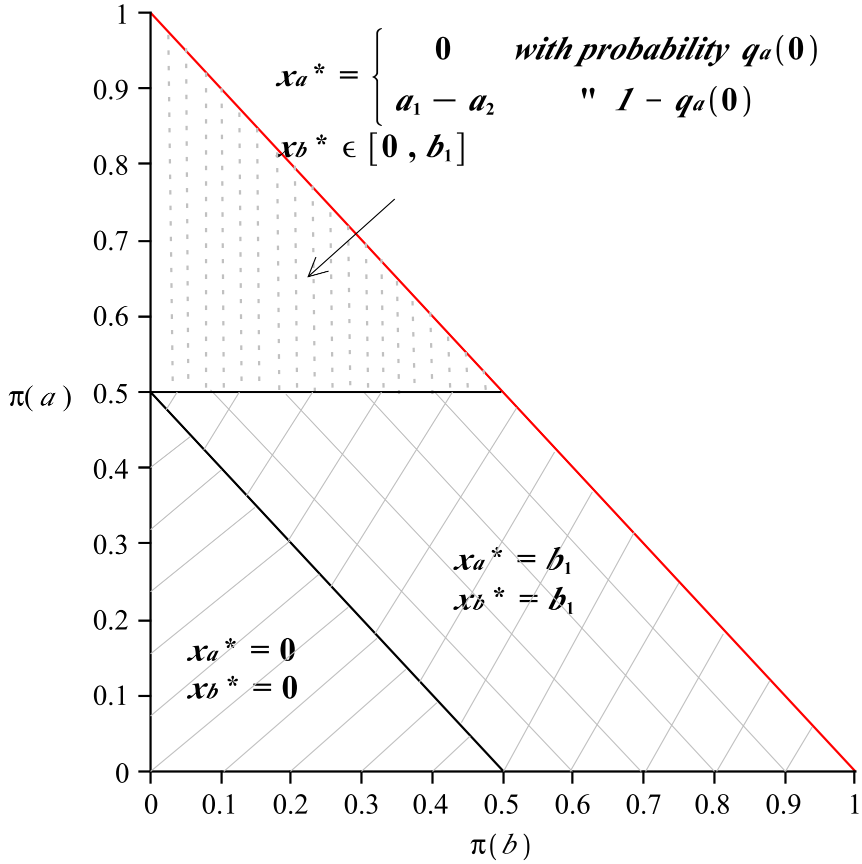

Figure 2: Continuum Equilibria for $b_1 < a_1 - a_2$

of the proof technique to be addressed.

4.3 Finite Replicated Markets and Equilibrium Convergence

In finite replicated markets, the analysis is made more difficult since the firms face aggregate uncertainty about the realization of firm types. In order to establish convergence of the equilibria in finite replicated markets to the continuum equilibrium, we need to first establish the existence for an n-replicated market, of a Bayesian Nash equilibrium in continuous, mixed (distributional) strategies with interval support.

4.3.1 Existence

As in both the two-firm, two-worker and the continuum cases, the equilibrium depends on the parameters of the model. We can break up the proof of existence into several cases. The proof is by construction. We consider firm types a and b, although things are symmetric for types c and d.

Proposition 4. *Given any finite replicated market with $2n$ firms, n workers in class W_1 and n workers in class W_2, there exists an equilibrium $(G_a^*(\cdot), G_b^*(\cdot), G_c^*(\cdot), G_d^*(\cdot))$ such that $G_k^*(\cdot)$ is a continuous distribution with interval support in the salary space, for all $k = a, b, c, d$. The equilibrium supports for types a and b satisfy*

$$\begin{aligned} 0 = \underline{x}_a < \overline{x}_a &= \underline{x}_b < \overline{x}_b \\ \textit{or} \quad 0 = \underline{x}_b < \overline{x}_b &= \underline{x}_a < \overline{x}_a. \end{aligned}$$

The analogous result holds for the equilibrium supports of types c and d.

Proof

See Appendix. ■

4.3.2 Convergence of Finite Market Equilibria

Next we show numerically that the replicated market equilibrium strategies converge point-wise to the corresponding continuum equilibrium as the number of replications goes to infinity. We return to Example 1 to demonstrate this result. Recall that $a = (2, 1)$, $b = (2, -1)$, $c = (1, 2)$, and $d = (-1, 2)$, while $\pi(a) = \frac{1}{2}$, $\pi(b) = \frac{1}{8}$, $\pi(c) = \frac{1}{4}$, and $\pi(d) = \frac{1}{8}$. The corresponding continuum equilibrium is as follows,

$$x_a^* = \begin{cases} 0 & \text{with probability } \frac{1}{4} \\ 1 & \text{with probability } \frac{3}{4} \end{cases} \tag{4}$$

$$x_b^* = 1 \tag{5}$$

$$x_c^* = 0 \tag{6}$$

$$x_d^* = 0. \tag{7}$$

Therefore, we need to show that the equilibrium distributions for types b, c, and d converge to degenerate distributions with all the mass placed on a single salary (1 for type b, 0 for types c and d), and that the equilibrium distribution for type a converges to a discrete distribution that places mass $\frac{1}{4}$ on 0 and mass $\frac{3}{4}$ on 1.

Now recall that the equilibrium distribution for a type a firm in an n-replicated market satisfies

$$
\begin{aligned}
\underline{x}_a &= 0 \\
\overline{x}_a &= (a_1 - a_2)\left[\sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(b)-\pi(a)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!}\right]
\end{aligned}
$$

and for all $x \in [\underline{x}_a, \overline{x}_a]$,

$$
\begin{aligned}
x &= (a_1 - a_2) \sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \left[\frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(a)-\pi(b)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!}\right. \\
&\quad \times \left.\sum_{t=0}^{n-j-1} \binom{k}{t} G_a^*(x)^{k-t} [1 - G_a^*(x)]^t\right].
\end{aligned}
$$

We can use the last equation to solve, given any value of $G_a^*(x)$, for the corresponding value of x. We calculated the pairs $\Big(x, G_a^*(x)\Big)$ that satisfy the indifference equations for several different values of n. If we plot these pairs with x on the horizontal axis and $G_a^*(x)$ on the vertical axis, we have a good representation of the equilibrium distribution for the different sized markets. We illustrate these in Figure 3.

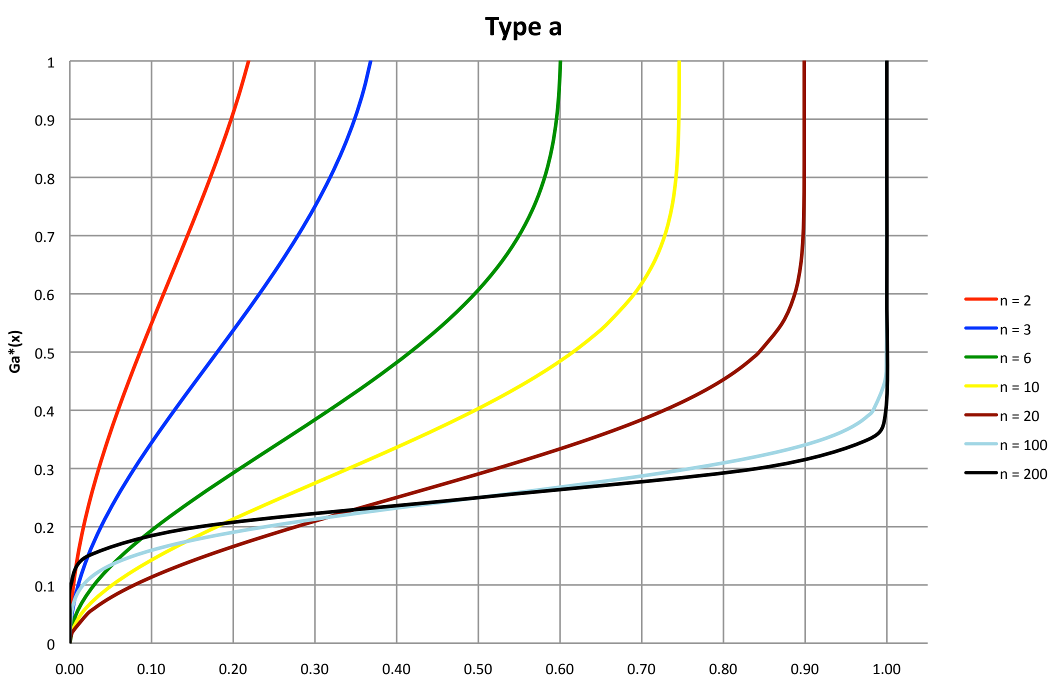

Figure 3: Replicated Market Equilibria for Type a

5 Conclusion

In this paper, we presented three main results. The first is that in any mixed strategy Bayesian Nash equilibrium with continuous strategies on interval supports, the equilibrium supports for types with a common most preferred worker meet at a common boundary. We call this our separation result. Our second result is a characterization of the mixed strategy, Bayesian Nash equilibria for a general two-firm, two-worker model with private information. We derive conditions on the relative marginal values of types with common most preferred workers that determine which of the types chooses the higher support.

There are several potential extensions on the simple two-firm, two-worker model. The most natural of these is to extend the model to a general n-firm, n-worker model. To ease the transition, we restrict the workers to be in one of two classes, denoted by W_1 and W_2. In order to examine larger environments, we consider replicated markets, using the two-firm, two-worker model as the baseline case. Our third result is a characterization of the equilibria for the limit case in which there are a continuum of firms and continuum of workers divided between the two classes. Finally, we introduce a conjecture about the convergence of finitely replicated market equilibria to the corresponding continuum equilibrium.

Another extension we might eventually consider is to allow for different proportions of the two worker classes in the n-firm, n-worker model. Then the competitive pressures on the salaries for different worker classes will depend not only on the relative 'popularity' of the workers, as determined by the type distribution, but also on the relative surplus or shortage of workers for each class. We also can extend the paper to relax the restriction that firms with the same preference ordering must have the same utility representation for that ordering. For instance, let each firm's type be a pair of values (x, y), each drawn independently from some interval $[\underline{\theta}, \overline{\theta}]$ according to a given distribution.

The two-firm, two-worker model, or indeed a small $n \times n$ model with just two worker classes, is both simple enough and interesting enough that it could form the basis for a series of experiments examining the strategic behavior of the firms. Although the model is designed to give firms a dominant strategy to make offers in order of preference, we would also have a chance to examine the sequence of offers chosen by the subject firms in an experiment. A body of experimental data might provide some indication of the strategies firms actually play when their types are private information.

Finally, it would be useful to examine how the theoretical results in this paper fit with empirical observations in different labor markets. In many labor markets, salaries do not accurately reflect differences in productivity. Instead, salaries tend to exhibit much less variation than worker productivities. Even though our model does not include explicit levels of productivity, a modified version may provide some explanation for this empirical trend in workers' salaries.

References

[1] Bulow, J. & J. Levin (2006), "Matching and Price Competition," *The American Economic Review*, 96(3), 652-668.

[2] Crawford, V. & E. Knoer (1981), "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49(2), 437-450.

[3] Gale, D. & L. Shapley (1962), "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9-15.

[4] Hatfield, J. & P. Milgrom (2005), "Matching with Contracts," *The American Economic Review* 95(4), 913-935.

[5] Kelso, A. & V. Crawford (1982), "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50(6), 1483-1504.

[6] Kojima, F. (2007), "Matching and Price Competition: Comment," *The American Economic Review*, 97(3), 1027-1031.

[7] Roth, A. (1982), "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7(4), 617-628.

[8] Roth, A. & M. Sotomayor (1990), "Two-sided matching: A study in game-theoretic modeling and analysis," Cambridge University Press, Cambridge.

[9] Shapley, L. & M. Shubik (1972), "The Assignment Game I: The Core," *International Journal of Game Theory*. 1, 111-130.

Appendix

Proof of Proposition 1

Case 1: $0 = \underline{x}_a < \underline{x}_b \leq \overline{x}_a < \overline{x}_b$

Consider the interval $[\underline{x}_b, \overline{x}_a]$, on which both firm types a and b make offers. Suppose by means of contradiction that this interval has a nonempty interior. For type a, the expected payoff for any salary in the interval is

$$\mathbb{E}U_a(x) = (a_1 - a_2)\Big[\pi(a)G_a^*(x) + \pi(b)G_b^*(x)\Big] - (a_1 - a_2)\Big[\pi(a) + \pi(b)\Big] + a_1 - x$$

for all $x \in [\underline{x}_b, \overline{x}_a]$. To make type a indifferent on the interval, we must have

$$g_a^*(x) = \frac{1}{\pi(a)(a_1 - a_2)} - \frac{\pi(b)}{\pi(a)} g_b^*(x), \qquad \forall x \in [\underline{x}_b, \overline{x}_a]. \tag{8}$$

Integrating equation (8) with respect to x yields

$$G_a^*(x) = G_a^*(\underline{x}_b) + \frac{x - \underline{x}_b}{\pi(a)(a_1 - a_2)} - \frac{\pi(b)}{\pi(a)} G_b^*(x), \tag{9}$$

for all $x \in [\underline{x}_b, \overline{x}_a]$.

On the other hand, type b has expected payoff

$$\mathbb{E}U_b(x) = (b_1 - x)\Big[\pi(a)G_a^*(x) + \pi(b)G_b^*(x) + \pi(c) + \pi(d)\Big], \qquad \forall x \in [\underline{x}_b, \overline{x}_a].$$

For type b to be indifferent on the interval, we need

$$g_b^*(x) - \frac{G_b^*(x)}{b_1 - x} = \frac{\pi(c) + \pi(d) + \pi(a)G_a^*(x)}{\pi(b)(b_1 - x)} - \frac{\pi(a)}{\pi(b)} g_a^*(x).$$

Solving this differential equation and using integration by parts, we find

$$G_b^*(x) = \frac{\eta(x) - \eta(\underline{x}_b)}{\pi(b)(b_1 - x)}, \qquad \forall x \in [\underline{x}_b, \overline{x}_a], \tag{10}$$

where $\eta(s) = s(\pi(c) + \pi(d)) - \pi(a)G_a^*(s)(b_1 - s)$. Then, by substituting equation (10) into equation (9) and simplifying, we must have

$$G_a^*(\underline{x}_b) = \frac{1}{\pi(a)}\left[\frac{b_1 - x}{a_1 - a_2} - \pi(c) - \pi(d)\right] \tag{11}$$

for **every** $x \in [\underline{x}_b, \overline{x}_a]$. Since $\pi(a) > 0$ and $a_1 > a_2$, the right hand side of equation (11) is strictly decreasing in x, which implies that the interior of the interval $[\underline{x}_b, \overline{x}_a]$ must be empty. That is, $\underline{x}_b = \overline{x}_a$.

Case 2: $0 = \underline{x}_b < \underline{x}_a \leq \overline{x}_b < \overline{x}_a$

Consider the interval $[\underline{x}_a, \overline{x}_b]$, on which both type a and type b make offers. Suppose again by contradiction that this interval has a nonempty interior. For type b, the expected payoff for any salary in the interval is

$$(b_1 - x)\Big[\pi(a)G_a^*(x) + \pi(b)G_b^*(x) + \pi(c) + \pi(d)\Big]$$

. To make type b indifferent on the interval, we must have

$$g_b^*(x) - \frac{G_b^*(x)}{b_1 - x} = \frac{\pi(c) + \pi(d) + \pi(a)G_a^*(x)}{\pi(b)(b_1 - x)} - \frac{\pi(a)}{\pi(b)}g_a^*(x). \tag{12}$$

Solving the differential equation (12) and using integration by parts to simplify the solution, we have

$$G_b^*(x) = \frac{(x - \underline{x}_a)(\pi(c) + \pi(d))}{\pi(b)(b_1 - x)} - \frac{\pi(a)}{\pi(b)}G_a^*(x) + \frac{k}{b_1 - x}, \qquad \forall x \in [\underline{x}_a, \overline{x}_b], \tag{13}$$

where k is some constant of integration. Using the fact that $G_b^*(\overline{x}_b) = 1$, we can solve for

$$k = b_1 - \overline{x}_b + \frac{\pi(a)}{\pi(b)}G_a^*(\overline{x}_b)(b_1 - \overline{x}_b) - \frac{\pi(c) + \pi(d)}{\pi(b)}(\overline{x}_b - \underline{x}_a). \tag{14}$$

Substituting (14) into (13) and simplifying gives

$$G_b^*(x) = \frac{b_1 - \overline{x}_b}{\pi(b)(b_1 - x)}\Big[\pi(b) + \pi(a)G_a^*(\overline{x}_b)\Big] - \frac{\pi(a)}{\pi(b)}G_a^*(x) - \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}(\overline{x}_b - x). \tag{15}$$

For type a, the expected payoff for any salary in the interval is

$$\mathbb{E}U_a(x) = (a_1 - a_2)\Big[\pi(a)G_a^*(x) + \pi(b)G_b^*(x)\Big] - (a_1 - a_2)\Big[\pi(a) + \pi(b)\Big] + a_1 - x,$$

for all $x \in [\underline{x}_a, \overline{x}_b]$. Solving to make type a indifferent and using the fact that $G_a^*(\underline{x}_a) = 0$ by assumption, we have

$$G_a^*(x) = \frac{x - \underline{x}_a}{\pi(a)(a_1 - a_2)} - \frac{\pi(b)}{\pi(a)}\Big[G_b^*(x) - G_b^*(\underline{x}_a)\Big]. \tag{16}$$

Substituting equation (15) into equation (16) and simplifying yields

$$\begin{aligned}\frac{x - \underline{x}_a}{a_1 - a_2} + \frac{[\pi(c) + \pi(d)](\overline{x}_b - x)}{b_1 - x} - \frac{b_1 - \overline{x}_b}{b_1 - x}(\pi(b) + \pi(a)G_a^*(\overline{x}_b) & \quad (17)\\ = \frac{[\pi(c) + \pi(d)](\overline{x}_b - \underline{x}_a)}{b_1 - \underline{x}_a} - \frac{b_1 - \overline{x}_b}{b_1 - \underline{x}_a}(\pi(b) + \pi(a)G_a^*(\overline{x}_b)) & \quad (18)\end{aligned}$$

for all $x \in [\underline{x}_a, \overline{x}_b]$. Notice that the right hand side of equation (17) is constant. To maintain equality, the derivative of the left hand side with respect to x must be 0 for **every** $x \in [\underline{x}_a, \overline{x}_b]$. However, this derivative,

$$\frac{1}{a_1 - a_2} - \frac{b_1 - \overline{x}_b}{(b_1 - x)^2}(\pi(c) + \pi(d) - \pi(b) - \pi(a)G_a^*(\overline{x}_b)), \tag{19}$$

changes with x unless $b_1 - \overline{x}_b = 0$ or $\pi(c) + \pi(d) = \pi(a)G_a^*(\overline{x}_b) + \pi(b)$. Furthermore, in those cases, the equation (19) equals $\frac{1}{a_1 - a_2} > 0$ since $a_1 > a_2$. It follows then that the interior of $[\underline{x}_a, \overline{x}_b]$ must be empty; that is, $\underline{x}_a = \overline{x}_b$.

Case 3: $0 = \underline{x}_b \leq \underline{x}_a < \overline{x}_a \leq \overline{x}_b$

Consider the interval $[\underline{x}_a, \overline{x}_a]$, on which both types make offers. As above, we can solve for

$$G_b^*(x) = \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}(x - \underline{x}_a) - \frac{\pi(a)}{\pi(b)}G_a^*(x), \tag{20}$$

for all $x \in [\underline{x}_a, \overline{x}_a]$, and

$$G_a^*(x) = \frac{x - \underline{x}_a}{\pi(a)(a_1 - a_2)} - \frac{\pi(b)}{\pi(a)}\Big[G_b^*(x) - G_b^*(\underline{x}_a)\Big], \tag{21}$$

for all $x \in [\underline{x}_a, \overline{x}_a]$. In order to satisfy both equation (20) and equation (21), we substitute the latter into the former and simplify, to obtain

$$G_b^*(\underline{x}_a) = \left[\frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)} - \frac{1}{b_1(a_1 - a_2)}\right](x - \underline{x}_a). \tag{22}$$

The left hand side of equation (22) is a constant. To maintain the equality, we must have the derivative of the right hand side equal to 0 for **every** $x \in [\underline{x}_a, \overline{x}_a]$, which means

$$\left[\frac{\pi(c)+\pi(d)}{\pi(b)(b_1-x)} - \frac{1}{b_1(a_1-a_2)}\right] = \frac{\pi(c)+\pi(d)}{\pi(b)(b_1-x)^2}(x-\underline{x}_a) \tag{23}$$

However, equation (23) has a unique solution, which implies that $\underline{x}_a = \overline{x}_a$. Since we have already ruled out pure strategies as best responses, it follows that there are no equilibria of the form described by **Case 3**.

<u>**Case 4:**</u> $0 = \underline{x}_a \leq \underline{x}_b < \overline{x}_b \leq \overline{x}_a$

The steps to prove that **Case 4** cannot occur are analogous to the steps for **Case 3**. We conclude that in order to simultaneously satisfy the conditions for indifference by both types on the interval $[\underline{x}_b, \overline{x}_b]$, we must have $\underline{x}_b = \overline{x}_b$, which we know cannot be true, since we have already ruled out pure strategies. Thus, there are no equilibria of the form described by **Case 4**. ■

Proof of Proposition 2

By Proposition 1, in any Bayesian Nash equilibrium, the supports for the strategies of two types with a common most preferred worker (say, a and b) must satisfy either **Case 1** with $\overline{x}_a = \underline{x}_b$, or **Case 2** with $\overline{x}_b = \underline{x}_a$.

Consider **Case 1** and suppose that $G_a^*(\cdot)$ and $G_b^*(\cdot)$ are the candidate equilibrium strategies for types a and b. For each salary x in the interval $[0, \overline{x}_a]$, type a firms have an expected payoff equal to

$$\mathbb{E}U_a(x) = a_1\pi(a)G_a^*(x) + a_2\pi(a)(1-G_a^*(x)) + a_2\pi(b)] + a_1[\pi(c)+\pi(d)] - x.$$

Since the firm needs to be indifferent between any salary that is offered as part of its equilibrium strategy, we must have

$$g_a^*(x) = \frac{1}{\pi(a)(a_1-a_2)} \qquad \forall x \in (0, \overline{x}_a]. \tag{24}$$

Integrating with respect to x yields

$$\begin{aligned} G_a^*(x) &= G_a^*(0) + \int_0^x \frac{1}{\pi(a)(a_1 - a_2)} ds & (25) \\ &= G_a^*(0) + \frac{x}{\pi(a)(a_1 - a_2)}, & (26) \end{aligned}$$

for all $x \in [0, \overline{x}_a]$. We assume that when both firms choose a salary of 0, the workers flip a coin if they have to decide between the two offers. As a result, the payoff from $x = 0$ is strictly less than from some small $\epsilon > 0$. Thus, $G_a^*(0) = 0$. Then we have

$$G_a^*(x) = \frac{x}{\pi(a)(a_1 - a_2)} \qquad \forall x \in [0, \overline{x}_a], \tag{27}$$

and since $G_a^*(\overline{x}_a) = 1$, we can solve for $\overline{x}_a = \pi(a)(a_1 - a_2)$.

Similarly, for each salary x in the interval $[\overline{x}_a, \overline{x}_b]$, type b firms have an expected payoff equal to

$$\mathbb{E}U_b(x) = (b_1 - x)[\pi(b)G_b^*(x) + 1 - \pi(b)].. \tag{28}$$

In order for type b firms to be indifferent between all the salaries in the interval $[\overline{x}_a, \overline{x}_b]$, we must have

$$g_b^*(x) - \frac{G_b^*(x)}{b_1 - x} = \frac{1 - \pi(b)}{\pi(b)(b_1 - x)}, \qquad \forall x \in (\overline{x}_a, \overline{x}_b] \tag{29}$$

Solving the differential equation in 29 gives

$$G_b^*(x) = \frac{1 - \pi(b)}{\pi(b)(b_1 - x)}(x - \overline{x}_a) + \frac{c}{b_1 - x}, \qquad \forall x \in [\overline{x}_a, \overline{x}_b]. \tag{30}$$

$G_b^*(\overline{x}_b) = 1$ allows us to solve for

$$c = b_1 - \overline{x}_b - \frac{1 - \pi(b)}{\pi(b)}(\overline{x}_b - \overline{x}_a),$$

and substitue into equation 30, which simplifies then to

$$G_b^*(x) = 1 - \frac{\overline{x}_b - x}{\pi(b)(b_1 - x)}, \qquad \forall x \in [\overline{x}_a, \overline{x}_b]. \tag{31}$$

Having solved for $\overline{x}_a = \pi(a)(a_1 - a_2)$, we use the fact that $G_b^*(\overline{x}_a) = 0$ to solve for

$$\overline{x}_b = \pi(b)b_1 + (1 - \pi(b))\pi(a)(a_1 - a_2). \tag{32}$$

Substituting equation 32 into equation 31 and simplifying gives the equilibrium strategy for type b firms,

$$\begin{aligned} G_b^*(x) \;&=\; \frac{1 - \pi(b)}{\pi(b)(b_1 - x)} \left[x - \pi(a)(a_1 - a_2)\right] \\ &\quad \text{on the support } \left[\pi(a)(a_1 - a_2),\, \pi(b)b_1 + (1 - \pi(b))\pi(a)(a_1 - a_2)\right] \end{aligned}$$

The condition that $b_1 > \pi(a)(a_1 - a_2)$ follows immediately, since if $b_1 < \pi(a)(a_1 - a_2)$, then $\left[\pi(a)(a_1 - a_2),\, \pi(b)b_1 + (1 - \pi(b))\pi(a)(a_1 - a_2)\right]$ is not an interval; the upper bound is less than the lower bound. This takes care of **Case 1**.

Now consider **Case 2**. For each salary x in the interval $[0, \overline{x}_b]$, type b firms have an expected payoff equal to

$$\mathbb{E}U_b(x) = (b_1 - x)[\pi(b)G_b^*(x) + \pi(c) + \pi(d)]. \tag{33}$$

In order for type b firms to be indifferent between all the salaries in the interval $[0, \overline{x}_b]$, we must have

$$g_b^*(x) - \frac{G_b^*(x)}{b_1 - x} = \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}, \qquad \forall x \in (0, \overline{x}_b]. \tag{34}$$

Solving the differential equation in 34 gives

$$G_b^*(x) = \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}x + \frac{c}{b_1 - x}, \qquad \forall x \in (0, \overline{x}_b]. \tag{35}$$

For the same reasons as above, we can easily verify that $G_b^*(0) = 0$, which implies $c = 0$, and therefore

$$G_b^*(x) = \frac{\pi(c) + \pi(d)}{\pi(b)(b_1 - x)}x \qquad \forall x \in [0, \overline{x}_b]. \tag{36}$$

Since $G_b^*(\overline{x}_b) = 1$, we can solve for $\overline{x}_b = \frac{\pi(b)b_1}{1 - \pi(a)}$.

For a type a firm, the expected payoff for each salary $x \in [\overline{x}_b, \overline{x}_a]$ is given by

$$\mathbb{E}U_a(x) = a_1\pi(a)G_a^*(x) + a_2\pi(a)(1 - G_a^*(x)) + a_1(1 - \pi(a)) - x. \tag{37}$$

For firm type a to be indifferent on the interval, we must have

$$g_a^*(x) = \frac{1}{\pi(a)(a_1 - a_2)}, \qquad \forall x \in [\overline{x}_b, \overline{x}_a]. \tag{38}$$

Integrating and using the fact that $G_a^*(\overline{x}_b) = 0$, we obtain

$$G_a^*(x) = \frac{x - \overline{x}_b}{\pi(a)(a_1 - a_2)}, \tag{39}$$

for all $x \in [\overline{x}_b, \overline{x}_a]$. Then substituting $\overline{x}_b = \frac{\pi(b)b_1}{1-\pi(a)}$ into equation 39 and simplifying gives the equilibrium strategy for type a firms,

$$G_a^*(x) = \frac{(1 - \pi(a))x - \pi(b)b_1}{\pi(a)(1 - \pi(a))(a_1 - a_2)} \tag{40}$$

$$\text{on the support } \left[\frac{\pi(b)b_1}{1 - \pi(a)}, \overline{x}_a\right]. \tag{41}$$

Finally, using $G_a^*(\overline{x}_a) = 1$ allows us to solve for $\overline{x}_a = \frac{\pi(b)b_1}{1-\pi(a)} + \pi(a)(a_1 - a_2)$. This takes care of **Case 2**. The proof for types c and d is identical, except for the notation. ■

Proof of Proposition 4

Case 1: Let $\pi(a) + \pi(b) \geq \frac{1}{2}$ and suppose $b_1 \geq a_1 - a_2$.

We conjecture the existence of a pair of equilibrium distributions $(G_a^*(\cdot), G_b^*(\cdot))$, with supports $[0, \overline{x}_a]$ and $[\overline{x}_a, \overline{x}_b]$, respectively. In order to prove that these are in fact equilibrium strategies, we first need to show indifference between each of the salaries in their corresponding equilibrium supports.

First, consider type a firms. For any type a firm f, the expected utility of a salary $x_f \in (0, \overline{x}_a)$ is

$$\mathbb{E}U_a(x_f) = (a_1 - a_2) \cdot \mathbb{P}r[\mu(f) \in W_1] + a_2 - x_f. \tag{42}$$

The probability $\mathbb{P}r[\mu(f) \in W_1]$ consists of two terms that capture, respectively,

(1) the probability that the actual number of type a firms and type b firms is less than or equal to n (the number of class W_1 workers), plus

(2) the probability that

   - the actual number of type a's and type b's is greater than n,
   - the number of type b's is less than n, and
   - the number of type a firms that choose $x > x_f$ is less than or equal to $n-$ the number of type b's.

In any other realization of types and salaries, the firm f is matched with a worker $w \in W_2$ and so receives a payoff of a_2.

If we let j denote the number of type b firms and k denote the number of type a firms out of the $2n-1$ other firms, then we can rewrite the first term of $\mathbb{P}r[\mu(f) \in W_1]$ as

$$\sum_{s=0}^{n-1} \binom{2n-1}{s} \Big[\pi(a) + \pi(b)\Big]^s \cdot \Big[1 - \pi(a) - \pi(b)\Big]^{2n-1-s},$$

and rewrite the second term of $\mathbb{P}r[\mu(f) \in W_1]$ as

$$\sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \Bigg[\frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(a)-\pi(b)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!} \times \sum_{t=0}^{n-j-1} \binom{k}{t} G_a^*(x_f)^{k-t}[1-G_a^*(x_f)]^t \Bigg].$$

At the lowest salary in type a's support, $x_f = 0$, the expected payoff is equal to

$$a_2 + (a_1 - a_2)\Bigg[\sum_{s=0}^{n-1} \binom{2n-1}{s} [\pi(a)+\pi(b)]^s [1-\pi(a)-\pi(b)]^{2n-1-s}\Bigg]. \tag{43}$$

Since the firm must be indifferent between all of the salaries in the support $[0, \overline{x}_a]$, we can also solve for the value of $\overline{x}_a$ by equating the expected payoffs from $x_f = 0$ and $x_f = \overline{x}_a$. This implies that

$$\overline{x}_a = (a_1 - a_2)\Bigg[\sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \frac{(2n-1)!\pi(b)^j \pi(a)^k [1-\pi(b)-\pi(a)]^{2n-1-j-k}}{j!k!(2n-1-j-k)!}\Bigg]. \tag{44}$$

Moreover, indifference implies that, for all $x_f \in (0, \overline{x}_a)$, we have

$$x_f = (a_1 - a_2) \sum_{j=0}^{n-1} \sum_{k=n+1-j}^{2n-1-j} \left[\frac{(2n-1)! \pi(b)^j \pi(a)^k [1 - \pi(a) - \pi(b)]^{2n-1-j-k}}{j! k! (2n-1-j-k)!} \right.$$
$$\left. \times \sum_{t=0}^{n-j-1} \binom{k}{t} G_a^*(x_f)^{k-t} [1 - G_a^*(x)]^t \right]. \tag{45}$$

The right-hand side of the equation is a continuous function of $G_a^*(x_f)$, which we have assumed is a continuous function of x_f. Since the left-hand side is strictly increasing in x_f, it follows that there exists a continuous function $G_a^*(x)$ that satisfies the equation.

Now consider type b firms. For any type b firm f, the expected utility of a salary $x_f \in (\overline{x}_a, \overline{x}_b)$ is

$$\mathbb{E}U_b(x_f) = (b_1 - x_f) \cdot \mathbb{P}r[\mu(f) \in W_1]. \tag{46}$$

Since type b firms don't care about class W_2 workers, they are either matched with a class W_1 worker, or remain unmatched. In this case, $\mathbb{P}r[\mu(f) \in W_1]$ consists of two different terms,

(1) the probability that there are no more than $n-1$ other type b firms,

(2) the probability that

- there are more than $n-1$ other type b firms, but
- the number of type b firms that choose $x > x_f$ is less than or equal to $n-1$.

Again we let j denote the number of type b firms, however now let k denote the number of type b firms that choose $x < x_f$. Then we can write the first term of $\mathbb{P}r[\mu(f) \in W_1]$ as

$$\sum_{j=0}^{n-1} \binom{2n-1}{j} \pi(b)^j [1 - \pi(b)]^{2n-1-j}, \tag{47}$$

and the second term as

$$\sum_{j=n}^{2n-1} \binom{2n-1}{j} \pi(b)^j [1 - \pi(b)]^{2n-1-j} \sum_{k=j-n+1}^{j} \binom{j}{k} G_b^*(x_f)^k [1 - G_b^*(x_f)]^{j-k}. \tag{48}$$

Furthermore, the second term is 0 for $x_f = \overline{x}_a$ (the lowest salary in type b's support), which means that the equilibrium expected payoff for a type b firm from any $x \in [\overline{x}_a, \overline{x}_b]$

must be

$$(b_1 - \overline{x}_a)\left[\sum_{j=0}^{n-1} \binom{2n-1}{j} \pi(b)^j [1-\pi(b)]^{2n-1-j}\right]. \tag{49}$$

At the top of type b's support, the probability that any number of other type b's choose $x < x_f = \overline{x}_b$ is 1. Thus, the expected payoff from choosing $x_f = \overline{x}_b$ is just $b_1 - \overline{x}_b$. In order to ensure indifference, we must have

$$\overline{x}_b = b_1 - (b_1 - \overline{x}_a)\left[\sum_{j=0}^{n-1} \binom{2n-1}{j} \pi(b)^j [1-\pi(b)]^{2n-1-j}\right]. \tag{50}$$

Finally, for all $x \in (\overline{x}_a, \overline{x}_b)$, $G_b^*(x)$ must satisfy

$$\frac{b_1 - \overline{x}_b}{b_1 - x} = \left[\sum_{j=0}^{n-1} \binom{2n-1}{j} \pi(b)^j [1-\pi(b)]^{2n-1-j} \right. \tag{51}$$
$$\left. + \sum_{j=n}^{2n-1} \binom{2n-1}{j} \pi(b)^j [1-\pi(b)]^{2n-1-j} \sum_{k=j-n+1}^{j} \binom{j}{k} G_b^*(x_f)^k [1-G_b^*(x_f)]^{j-k}\right].$$

■